

# ASYMPTOTICS FOR THE RADON TRANSFORM ON HYPERBOLIC SPACES

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ABSTRACT. Let  $G/H$  be a hyperbolic space over  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$ , and let  $K$  be a maximal compact subgroup of  $G$ . Let  $D$  denote a certain explicit invariant differential operator, such that the non-cuspidal discrete series belong to the kernel of  $D$ . For any  $L^2$ -Schwartz function  $f$  on  $G/H$ , we prove that the Abel transform  $\mathcal{A}(Df)$  of  $Df$  is a Schwartz function. This is an extension of a result established in [2] for  $K$ -finite and  $K \cap H$ -invariant functions.

## 1. INTRODUCTION

The *Radon transform*  $R$  on the hyperbolic spaces  $G/H$ ,

$$Rf = \int_{N^*} f(\cdot nH) dn,$$

where  $N^* \subset G$  is a certain unipotent subgroup, and the associated *Abel transform*  $\mathcal{A}$ , were introduced and studied in [1] and [2]. Generalizing Harish-Chandra's notion of cusp forms for real semisimple Lie groups, a discrete series is said to be *cuspidal* if it is annihilated by the Radon transform. In contrast with the Lie group case, however, *non-cuspidal* discrete series exist. For the projective hyperbolic spaces, these are precisely the spherical discrete series, but for some real non-projective hyperbolic spaces, there also exist non-spherical non-cuspidal discrete series.

Let  $\mathcal{C}^2(G/H)$  denote the space of  $L^2$ -Schwartz functions on  $G/H$ . Except for some boundary cases,  $\mathcal{A}$  maps  $\mathcal{C}^2(G/H)$  into Schwartz functions in the absence of non-cuspidal discrete series. On the other hand,  $\mathcal{A}f$  can be explicitly calculated for functions  $f$  belonging to the non-cuspidal discrete series. To complete the picture, we prove below that  $\mathcal{A}$  essentially maps the orthocomplement in  $\mathcal{C}^2(G/H)$  of the non-cuspidal discrete series into Schwartz functions. To be more precise, let  $\Delta_\rho = \Delta + \rho_q^2$ , where  $\Delta$  denotes the Laplace–Beltrami operator on  $G/H$ , and consider the  $G$ -invariant differential operator  $D = \Delta_\rho(\Delta_\rho - \lambda_1^2) \dots (\Delta_\rho - \lambda_r^2)$ , where  $\lambda_1, \dots, \lambda_r$  are the parameters of the non-cuspidal discrete series. Then  $\mathcal{A}(Df)$  is a Schwartz function. This extends our previous result, [2, Theorem 6.1 (vi)], valid only for the dense  $G$ -invariant subspace of  $\mathcal{C}^2(G/H)$  generated by the  $K$ -irreducible  $(K \cap H)$ -invariant functions, to all Schwartz functions.

In [2] we also considered the exceptional case corresponding to the Cayley numbers  $\mathbb{O}$ . We expect our new result to hold for this case as well, but we have not been through the rather cumbersome details.

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## 2. THE RADON TRANSFORM

In this section, we define the Radon transform and the Abel transform for the projective hyperbolic spaces over the classical fields  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  and  $\mathbb{H}$ . We have tried to keep the presentation and notation to a minimum, see [1] and [2] for further details (including results and proofs).

Let  $x \mapsto \bar{x}$  be the standard (anti-) involution of  $\mathbb{F}$ . Let  $p \geq 0$ ,  $q \geq 1$  be two integers, and consider the Hermitian form  $[\cdot, \cdot]$  on  $\mathbb{F}^{p+q+2}$  given by

$$[x, y] = x_1 \bar{y}_1 + \cdots + x_{p+1} \bar{y}_{p+1} - x_{p+2} \bar{y}_{p+2} - \cdots - x_{p+1+q+1} \bar{y}_{p+1+q+1}, \quad (x, y \in \mathbb{F}^{p+q+2}).$$

Let  $G = \mathrm{U}(p+1, q+1; \mathbb{F})$  denote the group of  $(p+q+2) \times (p+q+2)$  matrices over  $\mathbb{F}$  preserving  $[\cdot, \cdot]$ . Thus  $\mathrm{U}(p+1, q+1; \mathbb{R}) = \mathrm{O}(p+1, q+1)$ ,  $\mathrm{U}(p+1, q+1; \mathbb{C}) = \mathrm{U}(p+1, q+1)$  and  $\mathrm{U}(p+1, q+1; \mathbb{H}) = \mathrm{Sp}(p+1, q+1)$  in standard notation. Put  $\mathrm{U}(p; \mathbb{F}) = \mathrm{U}(p, 0; \mathbb{F})$ , and let  $K = \mathrm{U}(p+1; \mathbb{F}) \times \mathrm{U}(q+1; \mathbb{F})$  be the maximal compact subgroup of  $G$  fixed by the Cartan involution on  $G$ .

Let  $x_0 = (0, \dots, 0, 1)^T$ , where superscript  $T$  indicates transpose. Let  $H = \mathrm{U}(p+1, q; \mathbb{F}) \times \mathrm{U}(1; \mathbb{F})$  be the subgroup of  $G$  stabilizing the line  $\mathbb{F} \cdot x_0$  in  $\mathbb{F}^{p+q+2}$ . The reductive symmetric space  $G/H$  can be identified with the projective hyperbolic space  $\mathbb{X} = \mathbb{X}(p+1, q+1; \mathbb{F})$ ,

$$\mathbb{X} = \{z \in \mathbb{F}^{p+q+2} : [z, z] = -1\} / \sim,$$

where  $\sim$  is the equivalence relation  $z \sim zu$ ,  $u \in \mathbb{F}^*$ .

Let  $X_t$ , for  $t \in \mathbb{R}$ , denote the element in the Lie algebra  $\mathfrak{g}$  of  $G$  with value  $t$  in the  $(1, p+q+2)$ 'th and  $(p+q+2, 1)$ 'th matrix entries (the two opposite corners in the anti-diagonal), and 0 otherwise. Let  $\mathfrak{a}_q$  denote the Abelian subalgebra given by  $\{X_t \mid t \in \mathbb{R}\}$ , let  $a_t = \exp(X_t)$  denote the exponential of  $X_t$ , and also define  $A_q = \exp(\mathfrak{a}_q)$ .

Let (considered as row vectors)

$$u = (u_1, \dots, u_p) \in \mathbb{F}^p \quad \text{and} \quad v = (v_q, \dots, v_1) \in \mathbb{F}^q,$$

and let  $w \in \mathrm{Im} \mathbb{F}$  (i.e.,  $w = 0$  for  $\mathbb{F} = \mathbb{R}$ ). Define  $N_{u,v,w} \in \mathfrak{g}$  as the matrix given by

$$N_{u,v,w} = \begin{pmatrix} -w & u & v & w \\ -\bar{u}^T & 0 & 0 & \bar{u}^T \\ \bar{v}^T & 0 & 0 & -\bar{v}^T \\ -w & u & v & w \end{pmatrix}.$$

Then  $\exp(N_{u,v,w}) = I + N_{u,v,w} + 1/2 N_{u,v,w}^2$ , and a small calculation yields that

$$(1) \quad a_t \exp(N_{u,v,w}) \cdot x_0 = (\sinh t + 1/2 e^t(|u|^2 - |v|^2) + e^t w, \bar{u}^T; -\bar{v}^T, \cosh t + 1/2 e^t(|u|^2 - |v|^2) + e^t w)^T,$$

for any  $t \in \mathbb{R}$ .

Define the nilpotent subalgebra  $\mathfrak{n}^*$  as follows, for  $p \geq q$ ,

$$(2) \quad \mathfrak{n}^* = \{N_{u,v,w} : u = (-\bar{v}^r, u'), v \in \mathbb{F}^q, u' \in \mathbb{F}^{p-q}\},$$

and, for  $p < q$ ,

$$(3) \quad \mathfrak{n}^* = \{N_{u,v,w} : v = (-\bar{u}^r, v'), u \in \mathbb{F}^p, v' \in \mathbb{F}^{q-p}\},$$

where  $u^r, v^r$  means that the order of the indices is reversed. By abuse of notation, we leave out the superscript  $r$  in what follows.

We finally also define the following  $\rho$ -factors. Let  $d = \dim_{\mathbb{R}} \mathbb{F}$ , and let  $\rho_q = \frac{1}{2}(dp + dq + 2(d-1)) \in \mathbb{R}$ , and  $\rho_1 = \frac{1}{2}(|dp - dq| + 2(d-1)) \in \mathbb{R}$ .

Let  $N^* = \exp(\mathfrak{n}^*)$  denote the nilpotent subgroup generated by  $\mathfrak{n}^*$ . For functions  $f$  on  $G/H$ , we define, assuming convergence,

$$(4) \quad Rf(g) = \int_{N^*} f(gn^*H) dn^* \quad (g \in G).$$

Let  $f \in \mathcal{C}^2(G/H)$ , the space of  $L^2$ -Schwartz functions on  $G/H$ . From [1] and [2], we know that the Radon transform  $Rf$  is a smooth function. Also, the integral defining  $R$  converges uniformly on compact sets, and  $R$  is  $G$ - and  $\mathfrak{g}$ -equivariant.

We define the associated Abel transform  $\mathcal{A}$  by  $\mathcal{A}f(a) = a^{\rho_1} Rf(a)$ , for  $a \in A_q$ . We are mainly interested in the values of  $Rf$  and  $\mathcal{A}f$  on the elements  $a_s$ , and thus define  $Rf(s) = Rf(a_s)$ , and,

similarly,  $\mathcal{A}f(s) = \mathcal{A}f(a_s)$ , for  $s \in \mathbb{R}$ . Let  $\Delta$  denote the Laplace–Beltrami operator on  $G/H$ . Then, for  $f \in \mathcal{C}^2(G/H)$ ,

$$(5) \quad \mathcal{A}(\Delta f) = \left( \frac{d^2}{ds^2} - \rho_q^2 \right) \mathcal{A}f \quad (s \in \mathbb{R}).$$

Finally, for  $R > 0$ , let  $C_R^\infty(G/H)$  denote the subspace of smooth functions on  $G/H$  with support in the ( $K$ -invariant) ‘ball’  $\{ka_s \cdot x_0 \mid |s| \leq R\}$  of radius  $R$ . Similarly, let  $C_R^\infty(\mathbb{R})$  denote the subspace of smooth functions on  $\mathbb{R}$  with support in  $[-R, R]$ , and let  $\mathcal{S}(\mathbb{R})$  denote the Schwartz space on  $\mathbb{R}$ .

### 3. THE DISCRETE SERIES AND THE ABEL TRANSFORM

Let  $q > 1$ , or  $d > 1$ . The discrete series for the projective hyperbolic spaces can then be parametrized as

$$\{T_\lambda \mid \lambda = \frac{1}{2}(dq - dp) - 1 + \mu_\lambda > 0, \mu_\lambda \in 2\mathbb{Z}\},$$

see [1] and [2]. The spherical discrete series are given by the parameters  $\lambda$  for which  $\mu_\lambda \leq 0$ , including the ‘exceptional’ discrete series corresponding to  $\lambda > 0$  for which  $\mu_\lambda < 0$ .

For  $q = d = 1$ , the discrete series is parameterized by  $\lambda \in \mathbb{R} \setminus \{0\}$  such that  $|\lambda| + \rho_q \in 2\mathbb{Z}$ , and there are no spherical discrete series.

The parameters  $\lambda$  are, via the formula  $\Delta f = (\lambda^2 - \rho_q^2)f$ , related to the eigenvalues of  $\Delta$  acting on functions  $f$  in the corresponding representation space in  $L^2(G/H)$ .

We have a complete classification of the cuspidal and non-cuspidal discrete series for the projective hyperbolic spaces, also including information about the asymptotics of the Radon and Abel transforms:

**Theorem 1.** *Let  $G/H$  be a projective hyperbolic space over  $\mathbb{R}, \mathbb{C}, \mathbb{H}$ , with  $p \geq 0, q \geq 1$ .*

- (i) *If  $d(q - p) \leq 2$ , then all discrete series are cuspidal.*
- (ii) *If  $d(q - p) > 2$ , then non-cuspidal discrete series exists, given by the parameters  $\lambda > 0$  with  $\mu_\lambda \leq 0$ . More precisely, if  $0 \neq f \in \mathcal{C}^2(G/H)$  belongs to  $T_\lambda$ , then  $\mathcal{A}f(s) = Ce^{\lambda s}$ , with  $C \neq 0$ .*
- (iii)  *$T_\lambda$  is non-cuspidal if and only if  $T_\lambda$  is spherical.*
- (iv) *If  $p \geq q$ , and  $f \in C_R^\infty(G/H)$ , for  $R > 0$ , then  $\mathcal{A}f \in C_R^\infty(\mathbb{R})$ .*
- (v) *If  $d(q - p) \leq 1$ , and  $f \in \mathcal{C}^2(G/H)$ , then  $\mathcal{A}f \in \mathcal{S}(\mathbb{R})$ .*
- (vi) *Assume  $d(q - p) > 1$ . Let  $D$  be the  $G$ -invariant differential operator  $\Delta_\rho(\Delta_\rho - \lambda_1^2) \dots (\Delta_\rho - \lambda_r^2)$ , where  $\lambda_1, \dots, \lambda_r$  are the parameters of the non-cuspidal discrete series, and  $\Delta_\rho = \Delta + \rho_q^2$ . Then  $\mathcal{A}(Df) \in \mathcal{S}(\mathbb{R})$ , for  $f \in \mathcal{C}^2(G/H)$ .*

The above theorem is almost identical to [2, Theorem 6.1], except for item (vi), which was only proved for functions in the (dense)  $G$ -invariant subspace  $\mathcal{V}$  of  $\mathcal{C}^2(G/H)$  generated by the  $K$ -irreducible  $(K \cap H)$ -invariant functions. Additionally, [2, Theorem 6.1] furthermore included the exceptional case corresponding to the Cayley numbers  $\mathbb{O}$ .

Theorem 1 (including the reformulation of (vi)) also holds for the real non-projective spaces  $SO(p + 1, q + 1)_e / SO(p + 1, q)_e$ , except for item (iii), due to the existence of non-cuspidal non-spherical discrete series corresponding to negative and odd values of  $\mu_\lambda$  in the exceptional series, see [1, Section 5].

The conditions in (vi) essentially state that  $\mathcal{A}f$  is a Schwartz function if  $f$  is perpendicular to all non-cuspidal discrete series. The factor  $\Delta_\rho$ , however, seems to be necessary (except in the real case with  $q - p$  odd), even for the case  $d(q - p) = 2$ , where there are no non-cuspidal discrete series.

In the next section, we prove Theorem 1(vi).

### 4. PROOF OF THEOREM 1(VI)

First we note, following [2, Section 10], that the Schwartz decay conditions are satisfied near  $-\infty$  for  $\mathcal{A}(f)$ , and thus also for  $\mathcal{A}(Df)$ . This leaves us to study the Abel transform near  $+\infty$ .

Let  $f \in \mathcal{C}^2(G/H)$ , and write  $f[x] = f(gH)$ , where  $x = g \cdot x_0$ . From (1) and (3), we get

$$Rf(s) = \int_{N^*} f(a_s n^* H) dn^* = \int_{\mathbb{R}^{dq-dp} \times \mathbb{R}^{dp} \times \mathbb{R}^{d-1}} f[(\sinh s - 1/2e^s|v'|^2 + e^s w, u; -u, -v', \cosh s - 1/2e^s|v'|^2 + e^s w)] dv' du dw.$$

Let  $v' = |v'|\bar{v}$ ,  $v = -\sinh s + 1/2e^s|v'|^2$ , such that  $|v'|^2 = 1 + 2e^{-s}v - e^{-2s}$ , and  $\bar{w} = e^s w$ . Then,

$$Rf(s) = e^{-ds} \int_{-\sinh s}^{\infty} \int_{\mathbb{S}^{dq-dp-1} \times \mathbb{R}^{dp} \times \mathbb{R}^{d-1}} f[(\bar{w} - v, u; -u, -(1 + 2e^{-s}v - e^{-2s})^{1/2}\bar{v}, e^{-s} - v + \bar{w})] \times (1 + 2e^{-s}v - e^{-2s})^{(dq-dp)/2-1} dv d\bar{v} du d\bar{w},$$

where  $\mathbb{S}^r$  is the unit sphere in  $\mathbb{R}^r$ .

We will use the identification of  $\mathbb{X} = \mathbb{X}(p+1, q+1; \mathbb{F})$  with

$$\mathbb{X} = \{z \in \mathbb{F}^{p+q+2} : [z, z] < 0\} / \sim,$$

and identify a function  $f$  on  $\mathbb{X}$  with a homogeneous function on  $\{z \in \mathbb{F}^{p+q+2} : [z, z] < 0\}$  of degree zero.

We now identify  $\mathbb{F}^{p+q+2}$  with  $\mathbb{R}^{d(p+q+2)}$  such that the coordinates satisfy  $\operatorname{Re} z_j = x_{dj}$ , for  $j = 1, \dots, p+q+2$ . Consider the real hyperbolic space  $\tilde{\mathbb{X}} = \{z \in \mathbb{F}^{p+q+2} : [z, z] = -1\}$ . The group  $\tilde{G} = \operatorname{O}(d(p+1), d(q+1))$  acts transitively on  $\tilde{\mathbb{X}}$ . Let  $\tilde{K} = \operatorname{O}(d(p+1)) \times \operatorname{O}(d(q+1))$  denote the standard maximal compact subgroup of  $\tilde{G}$ . Let  $U(\tilde{\mathfrak{k}})$ , respectively  $U(\mathfrak{k})$ , denote the universal enveloping algebra of the Lie algebra  $\tilde{\mathfrak{k}}$  of  $\tilde{K}$ , respectively of the Lie algebra  $\mathfrak{k}$  of  $K$ .

**Lemma 2.** *Let  $U \in U(\tilde{\mathfrak{k}})$ , then  $U$  maps  $\mathcal{C}^2(G/H)$  into itself.*

*Proof.* The lemma is obvious for  $d = 1$ . So assume  $d > 1$ . We note that any element  $x \in \tilde{\mathbb{X}}$  can be written as  $x = ka \cdot x_0$ , where  $k \in K$ , and  $a = a_s$ ,  $s \geq 0$ . Let  $\tilde{H} = \operatorname{O}(d(p+1), d(q+1) - 1)$ , and let  $\tilde{\mathfrak{m}}$  denote the commutator of  $A_q$  in the Lie algebra of  $\tilde{K} \cap \tilde{H}$ . Then  $\tilde{\mathfrak{k}} = \mathfrak{k} + \tilde{\mathfrak{m}}$ .

Let  $U_k = \operatorname{Ad}(k)U$ , for  $k \in K$ , then  $Uf = (\operatorname{Ad}(k^{-1})U_k)f$ . By the Campbell–Baker–Hausdorff formula, there exists an element  $U_k^0 \in U(\mathfrak{k})$ , such that  $U_k = U_k^0$  modulo the left ideal generated by  $\tilde{\mathfrak{m}}$ . This implies that  $Uf[ka \cdot x_0] = (\operatorname{Ad}(k^{-1})U_k^0)f[ka \cdot x_0]$ . The map  $k \mapsto \operatorname{Ad}(k^{-1})U_k^0$  is continuous into a finite dimensional subspace of  $U(\mathfrak{k})$ , and we can write  $Uf[ka \cdot x_0] = (\operatorname{Ad}(k^{-1})U_k^0)f[ka \cdot x_0] = \sum_i u_i(k)U_i f[ka \cdot x_0]$ , for a finite set of elements  $U_i \in U(\mathfrak{k})$  and continuous coefficients  $u_i(k)$ . It follows that  $Uf$  is in  $\mathcal{C}^2(G/H)$ .  $\square$

Define for  $t = (t_1, t_2, t_3) \in \mathbb{R}^3$ , the auxiliary function

$$G_f(t_1, t_2, t_3) = \int_{\mathbb{S}^{dq-dp-1} \times \mathbb{R}^{dp} \times \mathbb{R}^{d-1}} f[(\bar{w} + t_1, u; -u, t_2\bar{v}, t_3 + \bar{w})] d\bar{v} du d\bar{w},$$

and, with the identification  $z = e^{-s}$ , define the function  $F(z) = e^{ds}Rf(s)$ . Then, since  $\sinh s = -(z - z^{-1})/2$ , we get

$$(6) \quad F(z) = \int_{(z-z^{-1})/2}^{\infty} G_f(-v, -(1+2zv-z^2)^{1/2}, z-v)(1+2zv-z^2)^{(dq-dp)/2-1} dv.$$

**Lemma 3.** *The function  $G_f$  is homogeneous of degree  $dp + d - 1$ , for  $t \in \{t \mid t_1^2 - t_2^2 - t_3^2 < 0\}$ ,  $G_f$  is even in  $t_2$ , and satisfies  $G_f(-t_1, t_2, -t_3) = G_f(t_1, t_2, t_3)$ .*

*Let  $X$  be the differential operator on  $\mathbb{R}^3$  given by  $t_3\partial/\partial t_2 - t_2\partial/\partial t_3$ . For all  $f \in \mathcal{C}^2(G/H)$ , and all  $k, N \in \mathbb{N}$ , there exists a constant  $C$ , such that*

$$|X^k G_f(t)| \leq C(t_2^2 + t_3^2)^{-d(q-p)/4} (1 + \log(t_2^2 + t_3^2))^{-N},$$

*for all  $t \in \{t \mid t_1^2 - t_2^2 - t_3^2 = -1\}$ .*

*Proof.* The first statement follows from the homogeneity of  $f$  and the definition of  $G_f$ .

As before we identify  $\mathbb{F}^{p+q+2}$  with  $\mathbb{R}^{d(p+q+2)}$ . Define, for  $i = d(1+2p)+1, \dots, d(1+p+q)$ , the differential operator  $D_i f[x] = x_{d(p+q+2)} \partial / \partial x_i f[x] - x_i \partial / \partial x_{d(p+q+2)} f[x]$ . This operator is defined by the left action of an element  $T_i$  in  $O(d(q+1))$  (with value 1 in the last entry of the  $i$ 'th row, value  $-1$  in the last entry of the  $i$ 'th column, and 0 otherwise), and Lemma 2 thus gives that  $D_i$  maps  $\mathcal{C}^2(G/H)$  into itself.

Let now  $\bar{v} = (v_{d(1+2p)+1}, \dots, v_{d(1+p+q)}) \in \mathbb{S}^{d(q-p)-1}$ . The operator

$$Y_{\bar{v}} = \sum_{i=2+2p}^{1+p+q} v_i D_i,$$

also maps  $\mathcal{C}^2(G/H)$  into itself, and

$$|Y_{\bar{v}} f[x]| \leq d(q-p) \max_i (|D_i f[x]|).$$

Applying the operator  $X$  to the integrand in the definition of  $G_f$ , we get

$$\begin{aligned} Xf[t_1, u; -u, t_2 \bar{v}, t_3] &= t_3 (\sum_{d(1+2p)+1}^{d(1+p+q)} \partial / \partial x_i f[\cdot] v_i - t_2 \partial / \partial x_{d(p+q+2)} f[\cdot]) \\ &= t_3 (\sum_{d(1+2p)+1}^{d(1+p+q)} \partial / \partial x_i f[\cdot] v_i) - t_2 (\sum_{d(1+2p)+1}^{d(1+p+q)} v_i^2) \partial / \partial x_{d(p+q+2)} f[\cdot] \\ &= Y_{\bar{v}} f[t_1, u; -u, t_2 \bar{v}, t_3]. \end{aligned}$$

The inequality for  $X^k f$  follows from repeated use of this formula and from the asymptotic estimates of functions in  $\mathcal{C}^2(G/H)$ .  $\square$

In particular, it follows that the function  $v \mapsto X^k G_f(-v, -1, -v)$  has the same parity as  $k$ .

**Lemma 4.** *Let  $k_0$  be the largest integer such that  $k_0 < (dq - dp)/2$ , and let  $\epsilon = (dq - dp)/2 - k_0$ . Define  $t = t(z, v) = (-v, -(1 + 2zv - z^2)^{1/2}, z - v)$ . Then*

- (i) *For  $k \leq k_0$ , the function  $v \mapsto \partial^k / \partial z^k (G_f(t(z, v)) (1 + 2zv - z^2)^{(dq-dp)/2-1})$  is uniformly integrable over  $\mathbb{R}$  for  $z < 1$ .*
- (ii) *For  $k \leq k_0$  odd, the function  $v \mapsto \partial^k / \partial z^k (G_f(t(z, v)) (1 + 2zv - z^2)^{(dq-dp)/2-1})$  is an odd function of  $v$  for  $z = 0$ .*

*Proof.* Notice that  $t_1^2 - t_2^2 - t_3^2 = -1$  and  $t_2^2 + t_3^2 = 1 + v^2$ , for  $t = t(z, v)$ , and that the integral (6) is uniformly convergent for  $0 \leq z \leq K < \infty$ . The same holds with  $G_f$  replaced by  $X^k G_f$ .

Repeated use of the formula  $\partial / \partial z (G_f(t(z, v)) (1 + 2zv - z^2)^\alpha) = -X G_f(t(z, v)) (1 + 2zv - z^2)^{\alpha-1/2} + 2\alpha G_f(t(z, v)) (1 + 2zv - z^2)^{\alpha-1} (z - v)$  yields (i), and together with the parity properties of  $X^k G_f$  also gives (ii).  $\square$

We notice that  $\epsilon = 1$  if  $d(q-p)$  is even, and  $\epsilon = 1/2$  if  $d(q-p)$  is odd, i.e., if  $d = 1$  and  $q-p$  is odd.

For  $k < k_0$ , the derivatives  $\partial^k / \partial z^k$  of  $G_f(t(z, v)) (1 + 2zv - z^2)^{(dq-dp)/2-1}$  are zero at  $v = -\sinh s = \frac{1}{2}(z - z^{-1})$ , whence the integrand is at least  $k_0$  times differentiable near  $z = 0$ , and we can compute the derivatives  $d^k / dz^k F(z)$  by differentiating under the integral sign in (6).

If  $k_0 > 0$ , we can use Taylors formula to express  $F(z)$  as a polynomial of degree  $k_0 - 1$ , plus a remainder term involving  $d^{k_0} / dz^{k_0} F(\xi)$ , for some  $0 < \xi(z) < z$ ,

$$F(z) = c_0 + c_1 z + c_2 z^2 + \dots + c_{k_0-1} z^{k_0-1} + R_{k_0}(\xi) z^{k_0},$$

where  $0 < \xi < z$ , and

$$c_j = \frac{1}{j!} \int_{-\infty}^{\infty} \frac{d^j}{dz^j} \Big|_{z=0} (G_f(t(z, v)) (1 + 2zv - z^2)^{(dq-dp)/2-1}) dv,$$

for  $j \in \{0, \dots, k_0 - 1\}$ . The remainder term is given by:

$$R_{k_0}(\xi) = \frac{1}{k_0!} \int_{(\xi-\xi^{-1})/2}^{\infty} \frac{d^{k_0}}{dz^{k_0}} \Big|_{z=\xi} (G_f(t(z, v)) (1 + 2zv - z^2)^{(dq-dp)/2-1}) dv.$$

Consider  $\mathcal{A}f(s) = e^{\rho_1 s} Rf(s) = z^{-(\rho_1-d)} F(z)$ , which is equal to

$$c_0 z^{-(\rho_1-d)} + c_1 z^{-(\rho_1-d-1)} + c_2 z^{-(\rho_1-d-2)} + \dots + c_{k_0-1} z^{-\epsilon} + z^{(-\epsilon+1)} R_{k_0}(\xi).$$

Here we have used that  $\rho_1 - d = d(q-p)/2 - 1$ . For  $j$  even, the exponents  $-d(q-p)/2 - 1 - j$ , for  $j \in \{0, \dots, k_0 - 1\}$ , correspond to the parameters  $\lambda_1, \dots, \lambda_r$  for the non-cuspidal discrete series, and  $c_j = 0$  for  $j$  odd, since the integrand is an odd function.

For the real non-projective hyperbolic spaces the condition concerning the parity  $j$  does not hold, but in that case *all* the exponents  $-d(q-p)/2 - 1 - j$ , for  $j \in \{0, \dots, k_0 - 1\}$ , correspond to parameters  $\lambda_1, \dots, \lambda_r$  for the non-cuspidal discrete series, see [1, Section 3]

From the definition of the differential operator  $D$  and (5), we see that  $\mathcal{A}(Df)$  at most has a contribution from the remainder term, and further that  $\mathcal{A}(Df)$  does not have a constant term at  $\infty$ , due to the term  $d^2/ds^2$ . If  $\epsilon = 1/2$ , the remainder term  $e^{-1/2s} R_{k_0}(\xi(s))$  is clearly rapidly decreasing, and we are thus left to consider the case  $\epsilon = 1$ , in which case  $k_0 = d(q-p)/2 - 1$ .

Consider the constant term  $C_{R_{k_0}} = \lim_{s \rightarrow \infty} R_{k_0}(e^{-s})$ , which could be non-zero. We want to show that  $R_{k_0}(\xi) - C_{R_{k_0}}$  is rapidly decreasing at  $+\infty$ , where  $\xi = \xi(s)$ , with  $0 < \xi < e^{-s}$ . We also include the case  $k_0 = 0$ , where we put  $\xi = e^{-s}$ .

Define  $H(z, v) = \frac{d^{k_0}}{dz^{k_0}} (G_f(t(z, v))(1 + 2zv - z^2)^{k_0})$ . Then, for  $\xi < z < 1$ ,

$$R_{k_0}(\xi) - C_{R_{k_0}} = \int_{(\xi-\xi^{-1})/2}^{\infty} (H(\xi, v) - H(0, v)) dv + \int_{-\infty}^{(\xi-\xi^{-1})/2} H(0, v) dv = I_1(\xi) + I_2(\xi).$$

For  $I_1(\xi)$ , there exists  $\xi_1 = \xi_1(\xi, v) < \xi$ , such that  $H(\xi, v) - H(0, v) = \xi d/dz|_{z=\xi_1} H(z, v)$ , and we get:

$$I_1(\xi) < z \int_{-\infty}^{\infty} \left| \frac{d}{dz} \right|_{z=\xi_1} H(z, v) dv.$$

By Lemma 4, the integrand is uniformly integrable for  $z < 1$ , and we conclude that  $I_1(\xi)$  is bounded by  $Ce^{-s}$ .

For  $s$  large, the function  $H(0, v)$  is for every  $N \in \mathbb{N}$  bounded by

$$|H(0, v)| \leq C(1 + v^2)^{-(-d(q-p)/4)} |v|^{k_0} \log(1 + v^2)^{-N},$$

for some positive constant  $C$ . Using this, we find that

$$I_2(z) < C \int_{\sinh s}^{\infty} v^{-1} (\log(v))^{-N} dv = C(N-1)^{-1} (\log(\sinh s))^{-N+1} \leq Cs^{-N+1}.$$

It follows that  $R_{k_0}(\xi) - C_{R_{k_0}}$  is rapidly decreasing at  $+\infty$ , whence  $\mathcal{A}(Df)$  is rapidly decreasing at  $+\infty$ , which finishes the proof of Theorem 1.

## REFERENCES

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